

THE LEE–YANG AND PÓLYA–SCHUR PROGRAMS. III. ZERO-PRESERVERS ON BARGMANN–FOCK SPACES

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ABSTRACT. We characterize linear operators preserving zero-restrictions on entire functions in weighted Bargmann–Fock spaces. The characterization extends the main results of [2, 3, 4] to the realm of entire functions, and translates into an optimal, albeit formal, Lee–Yang theorem.

1. INTRODUCTION

The problem of describing linear operators preserving zero-restrictions on polynomials and transcendental entire functions has since the pioneering work of Hermite, Laguerre, Jensen and Pólya been revisited frequently, see e.g. [2, 3, 5, 7, 8]. Already in the seminal papers [6, 14] of Lee and Yang it was made evident that linear operators preserving zero restrictions play a prominent role in understanding phase transitions of spin models in statistical physics. The method of using such preservers was further developed by Lieb and Sokal [8] to prove a general Lee–Yang theorem. In a series of papers [2, 3, 4] joint with Borcea we have characterized linear operators on polynomials preserving the property of being non-vanishing whenever the variables are in prescribed open circular regions. This constitutes a vast generalization of Pólya and Schur’s theorem [10] characterizing diagonal linear operators on polynomials preserving real-rootedness. In this paper we extend the main results of [2, 3, 4] to weighted Bargmann–Fock spaces of entire functions. The extension makes the connection between the Pólya–Schur and Lee–Yang programs truly transparent. Indeed, our characterization (Theorem 3.1) of Laguerre–Pólya preservers translates directly into an optimal, albeit formal, Lee–Yang theorem (Theorem 4.5).

2. LAGUERRE–PÓLYA PRESERVERS

We say that a polynomial $P(z) \in \mathbb{C}[z_1, \dots, z_n]$ is *stable* if $P(z) \neq 0$ whenever $z \in H^n$ where $H = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, and that an entire function $f(z)$ in n variables is in the *complex Laguerre–Pólya class*, $f(z) \in \mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C})$, if f is the uniform limit on compact subsets of \mathbb{C}^n of stable polynomials. The (real) *Laguerre–Pólya class*, $\mathcal{L}\text{-}\mathcal{P}_n(\mathbb{R})$, consists of those functions in $\mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C})$ with real coefficients. Laguerre and Pólya proved that a univariate entire function is in

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the Laguerre–Pólya class if and only if it may be expressed as

$$f(z) = Cz^n e^{az-bz^2} \prod_{k=1}^{\omega} (1+x_k z) e^{-x_k z}, \quad (2.1)$$

where $C, a, x_k \in \mathbb{R}$ for all k , and $b \geq 0$, $\omega \in \mathbb{N} \cup \{\infty\}$, $n \in \mathbb{N}$ and $\sum_k x_k^2 < \infty$.

The symbol of a linear operator $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[z_1, \dots, z_m]$ is the formal power series

$$G_T(z, w) = T(e^{z \cdot w}) := \sum_{\alpha \in \mathbb{N}^n} T(z^\alpha) \frac{w^\alpha}{\alpha!},$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$ and $z \cdot w = z_1 w_1 + \cdots + z_n w_n$. We say that T *preserves stability* if $T(P)$ is stable or identically zero whenever P is stable. The following characterizations of stability preservers was achieved in [3].

Theorem 2.1. *Let $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[z_1, \dots, z_m]$ be a linear operator. Then T preserves stability if and only if*

- (1) *The rank of T is at most one and T is of the form*

$$T(P) = \alpha(P)Q,$$

where $\alpha : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}$ is a linear functional and Q is a stable polynomial, or

- (2) *$G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C})$, where $-w = (-w_1, \dots, -w_n)$.*

Theorem 2.2. *Let $T : \mathbb{R}[z_1, \dots, z_n] \rightarrow \mathbb{R}[z_1, \dots, z_m]$ be a linear operator. Then T preserves real stability if and only if*

- (1) *The rank of T is at most two and T is of the form*

$$T(P) = \alpha(P)Q + \beta(P)R,$$

where $\alpha, \beta : \mathbb{R}[z_1, \dots, z_n] \rightarrow \mathbb{R}$ are linear functionals and $Q + iR$ is a stable polynomial, or

- (2) *$G_T(z, w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{R})$, or*
(3) *$G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{R})$.*

Since a real univariate polynomial is stable if and only if it has only real zeros, Theorem 2.2 characterizes real zero preservers when $n = m = 1$.

We want to extend linear stability preservers to act on entire functions. More precisely we want entire functions in the (complex) Laguerre–Pólya class to be mapped on entire functions in the (complex) Laguerre–Pólya class. To achieve this we should at least demand that stable polynomials should be mapped into $\mathcal{L}\text{-}\mathcal{P}_m(\mathbb{C})$. However we shall see that this weakest requirement still allows us to extend the domain to classes of entire functions of bounded growth. Let $\mathbb{K}[[z_1, \dots, z_n]]$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , be the space of all formal power series with coefficients in \mathbb{K} . A \mathbb{K} -linear operator $T : \mathbb{K}[z_1, \dots, z_n] \rightarrow \mathbb{K}[[z_1, \dots, z_m]]$ is called *Laguerre–Pólya preserver* if

$$T(\mathcal{L}\text{-}\mathcal{P}_n(\mathbb{K}) \cap \mathbb{K}[z_1, \dots, z_n]) \subseteq \mathcal{L}\text{-}\mathcal{P}_m(\mathbb{K}),$$

that is, if stable polynomials are mapped into the (complex) Laguerre–Pólya class. The symbol of a linear operator $T : \mathbb{K}[z_1, \dots, z_n] \rightarrow \mathbb{K}[[z_1, \dots, z_m]]$ is the formal power series

$$G_T(z, w) = T(e^{z \cdot w}) := \sum_{\alpha \in \mathbb{N}^n} T(z^\alpha) \frac{w^\alpha}{\alpha!}.$$

Theorems 2.1 and 2.1 extend naturally to this general setting.

Theorem 2.3. *Let $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[[z_1, \dots, z_m]]$ be a linear operator. Then T is a Laguerre-Pólya preserver if and only if*

- (1) *The rank of T is at most one and T is of the form*

$$T(P) = \alpha(P)f,$$

where $\alpha : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}$ is a linear functional and $f \in \mathcal{L}\text{-}\mathcal{P}_m(\mathbb{C})$, or

- (2) *$G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C})$.*

Lemma 2.4. *Let $f(z, w)$ be a formal power series in $z_1, \dots, z_m, w_1, \dots, w_n$. Write f as*

$$f(z, w) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha(z) w^\alpha.$$

Then $f \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C})$ if and only if for each $\beta \in \mathbb{N}^n$

$$\Lambda_\beta(f) := \sum_{\alpha \in \mathbb{N}^n} (\beta)_\alpha a_\alpha(z) w^\alpha \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C}),$$

where

$$(\beta)_\alpha := \alpha! \prod_{i=1}^m \binom{\beta_i}{\alpha_i}.$$

Proof. The lemma was proved in [3, Theorem 6.1] in the case when a_α is a polynomial for all α . However the lemma follows from the special case. By [3, Theorem 6.1]

$$\begin{aligned} f(z, w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C}) &\iff \Lambda_{\gamma \oplus \beta}(f) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C}) \text{ for all } \gamma \in \mathbb{N}^m, \beta \in \mathbb{N}^n \\ &\iff \Lambda_\beta(f) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{C}) \text{ for all } \beta \in \mathbb{N}^n. \end{aligned}$$

□

Proof of Theorem 2.3. For $\beta \in \mathbb{N}^m$ let $T_\beta : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[z_1, \dots, z_m]$ be defined by $T_\beta = \Lambda_\beta \circ T$. Clearly T has rank at most one if and only if T_β has rank at most one for all β with $\min_{1 \leq i \leq n} \beta_i$ sufficiently large. By Lemma 2.4 T is a Laguerre-Pólya preserver if and only if T_β preserves stability for all $\beta \in \mathbb{N}^m$. Since $G_{T_\beta}(z, w) = \Lambda_\beta(G_T)$, where Λ_β acts on the z -variables, the theorem follows from Theorem 2.1 and Lemma 2.4. □

The proof of the real version of Theorem 2.3 follows similarly, and is therefore omitted.

Theorem 2.5. *Let $T : \mathbb{R}[z_1, \dots, z_n] \rightarrow \mathbb{R}[[z_1, \dots, z_m]]$ be a linear operator. Then T is a Laguerre-Pólya preserver if and only if*

- (1) *The rank of T is at most two and T is of the form*

$$T(P) = \alpha(P)f + \beta(P)g,$$

where $\alpha, \beta : \mathbb{R}[z_1, \dots, z_n] \rightarrow \mathbb{R}$ are linear functionals and $f + ig \in \mathcal{L}\text{-}\mathcal{P}_m(\mathbb{C})$, or

- (2) *$G_T(z, w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{R})$, or*
 (3) *$G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{m+n}(\mathbb{R})$.*

Remark 2.6. If $T : \mathbb{K}[z_1, \dots, z_n] \rightarrow \mathbb{K}[[z_1, \dots, z_n]]$ is a linear operator we may define the *formal adjoint*, $T^\# : \mathbb{K}[[z_1, \dots, z_n]] \rightarrow \mathbb{K}[[z_1, \dots, z_n]]$, as the linear operator with symbol $G_{T^\#}(z, w) = \overline{G_T(\overline{w}, \overline{z})}$. Since $(v, w) \in H \times (-H)$ if and only if $(\overline{w}, \overline{v}) \in H \times (-H)$ we see that T is a Laguerre–Pólya preserver if and only if $T^\#$ is a Laguerre–Pólya preserver (provided that T is of rank greater than one if $\mathbb{K} = \mathbb{C}$, and greater than two if $\mathbb{K} = \mathbb{R}$). This duality can be seen as a vast generalization of a famous theorem due to Hermite, Jensen, Pólya and Poulain: Let T be a formal differential operator with constant coefficients of the form $T = g(d/dz)$ where g is a real formal power series. Then T preserves real-rootedness if and only if $g \in \mathcal{L}\text{-}\mathcal{P}_1(\mathbb{R})$. The formal adjoint of T is the operator defined by $T^\#(f) = g(z)f(z)$, and so $T^\#$ is a Laguerre–Pólya preserver if and only if $g \in \mathcal{L}\text{-}\mathcal{P}_1(\mathbb{R})$.

In the next section we shall see that the formal adjoints considered here are actually proper adjoints in Hilbert spaces.

3. PRESERVERS ON WEIGHTED BARGMANN–FOCK SPACES

A priori, the linear operators in Theorems 2.3 and 2.5 may only be applied to polynomials. However we shall see that the domain extends naturally to entire functions of bounded growth. We want to find conditions on $G_T(z, w)$ that allow us to extend the domain to spaces of entire functions. It follows from [3, Theorem 6.6] that for each $f \in \mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C})$ there are constants $A, B > 0$ such that

$$|f(z)| \leq Ae^{Br^2}, \quad \text{whenever } |z_j| \leq r \text{ for all } 1 \leq j \leq n. \quad (3.1)$$

Hence functions in the Laguerre–Pólya class are of order at most two and of bounded type. Lieb and Sokal [8] worked with certain Frechét spaces of entire functions, we find it more convenient to work with Hilbert spaces: For $\beta \in \mathbb{R}_+^n := (0, \infty)^n$, define the β -weighted Bargmann–Fock space¹, \mathcal{F}_β , to be the space of all entire functions $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ such that

$$\|f\|_\beta^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!}{\beta^\alpha} |a_\alpha|^2 < \infty,$$

see [1]. One may also write

$$\|f\|_\beta^2 = \int_{\mathbb{C}^n} |f(z)|^2 d\sigma_\beta(z) := \frac{\beta_1 \cdots \beta_n}{\pi^n} \int_{\mathbb{C}^n} |f(z)|^2 \exp\left(-\sum_{i=1}^n \beta_i |z_i|^2\right) dm$$

where m is Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. By (3.1) we see that each $f \in \mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C})$ is in \mathcal{F}_β for some $\beta \in \mathbb{R}_+^n$. To be more precise if the entire function f is $\mathcal{O}(\exp(\beta_1 |z_1|^2/2 + \cdots + \beta_n |z_n|^2/2))$, then $f \in \mathcal{F}_\gamma$ for all $\gamma \gg \beta$ (by which we mean $\gamma_j > \beta_j$ for all $1 \leq j \leq n$). The space \mathcal{F}_β is a Hilbert space with inner product given by

$$\langle f, g \rangle_\beta = \sum_{\alpha \in \mathbb{N}^n} \frac{\alpha!}{\beta^\alpha} a_\alpha \overline{b_\alpha} = \int_{\mathbb{C}^n} f(z) \overline{g(z)} d\sigma_\beta(z),$$

and orthonormal basis

$$\left\{ \sqrt{\frac{\beta^\alpha}{\alpha!}} z^\alpha \right\}_{\alpha \in \mathbb{N}^n}.$$

¹Bargmann–Fock spaces have many names, usually a combination of Bargmann, Fischer, Fock and Segal.

It has a *reproducing kernel* given by $e_\beta(z, \bar{w}) := \exp\left(\sum_{j=1}^n \beta_j z_j \bar{w}_j\right)$, that is,

$$f(w) = \langle f(z), e_\beta(z, \bar{w}) \rangle_\beta, \quad (3.2)$$

for all $f \in \mathcal{F}_\beta$. In particular, by Cauchy–Schwartz inequality,

$$\begin{aligned} |f(w)|^2 &= |\langle f(z), e_\beta(z, \bar{w}) \rangle_\beta|^2 \leq \|f\|_\beta^2 \|e_\beta(z, \bar{w})\|_\beta^2 \\ &\leq \|f\|_\beta^2 \frac{\beta_1 \cdots \beta_n}{\pi^n} \int_{\mathbb{C}^n} \exp\left(-\sum_{i=1}^n \beta_i (|z_i|^2 - 2|z_i||w_i|)\right) dm = C(|w_1|, \dots, |w_n|) \|f\|_\beta^2, \end{aligned} \quad (3.3)$$

and hence convergence in $\|\cdot\|_\beta$ implies uniform convergence on compact subsets of \mathbb{C}^n . If $\alpha \in \mathbb{R}_+^m$ and $\beta \in \mathbb{R}_+^n$, let $\alpha \oplus \beta = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$, $\alpha^{-1} = (\alpha_1^{-1}, \dots, \alpha_m^{-1})$, and if $n = m$ let $\alpha\beta = (\alpha_1\beta_1, \dots, \alpha_n\beta_n)$.

Let $\mathcal{L}\text{-}\mathcal{P}_\beta(\mathbb{C}) = \mathcal{L}\text{-}\mathcal{P}_n(\mathbb{C}) \cap \mathcal{F}_\beta$. The following theorem tells us to which weighted Bargmann–Fock spaces a Laguerre–Pólya preserver may be extended. Theorem 3.1 (1) is sharp and the converse is given by the last sentence of Theorem 3.4.

Theorem 3.1. *Let $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[[z_1, \dots, z_m]]$ be a linear operator of rank at least two. Then T is a Laguerre–Pólya preserver if and only if $G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{\beta \oplus \gamma}(\mathbb{C})$ for some $\beta \in \mathbb{R}_+^n$ and $\gamma \in \mathbb{R}_+^m$.*

Moreover if $G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{\beta \oplus \gamma}(\mathbb{C})$, then

- (1) *T extends to a bounded linear operator $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ of the form (3.4) and (3.5) for all $\alpha \leq \gamma^{-1}$, and*
- (2) *$T : \mathcal{L}\text{-}\mathcal{P}_\alpha(\mathbb{C}) \rightarrow \mathcal{L}\text{-}\mathcal{P}_\beta(\mathbb{C})$, for all $\alpha \leq \gamma^{-1}$.*

The real version of Theorem 3.1 is similar. In (1) below one may either consider T as the obvious complexification of T , or consider T as a linear operator on the real weighted Bargmann–Fock space.

Theorem 3.2. *Let $T : \mathbb{R}[z_1, \dots, z_n] \rightarrow \mathbb{R}[[z_1, \dots, z_m]]$ be a linear operator of rank at least three. Then T is a Laguerre–Pólya preserver if and only if $G_T(z, w) \in \mathcal{L}\text{-}\mathcal{P}_{\beta \oplus \gamma}(\mathbb{R})$ or $G_T(z, -w) \in \mathcal{L}\text{-}\mathcal{P}_{\beta \oplus \gamma}(\mathbb{R})$ for some $\beta \in \mathbb{R}_+^n$ and $\gamma \in \mathbb{R}_+^m$.*

Moreover if $G_T(z, \pm w) \in \mathcal{L}\text{-}\mathcal{P}_{\beta \oplus \gamma}(\mathbb{R})$, then

- (1) *T extends to a bounded linear operator $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ of the form (3.4) and (3.5) for all $\alpha \leq \gamma^{-1}$, and*
- (2) *$T : \mathcal{L}\text{-}\mathcal{P}_\alpha(\mathbb{R}) \rightarrow \mathcal{L}\text{-}\mathcal{P}_\beta(\mathbb{R})$, for all $\alpha \leq \gamma^{-1}$.*

Example 3.3. For $n = 1$ and $\mathbb{K} = \mathbb{R}$, Laguerre–Pólya preservers are linear operators that send polynomials with only real zeros into the Laguerre–Pólya class. If the symbol of T is in $\mathcal{F}_{\beta \oplus \gamma}$, then T extends to all f as in (2.1) with $2b < 1/\gamma$. If T is a multiplier sequence, see [5, 10], then its symbol is of the form

$$G_T(z, w) = Cz^n w^n e^{\pm a zw} \prod_{j=1}^{\omega} (1 \pm x_j zw),$$

where $C \in \mathbb{R}$, $a \geq 0$, $n \in \mathbb{N}$, $\omega \in \mathbb{N} \cup \{\infty\}$, $x_j > 0$ for all $j \in \mathbb{N}$ and $\sum_j x_j < \infty$. Since

$$\prod_{j=1}^{\omega} (1 + x_j |z||w|) \leq \exp\left(|z||w| \sum_{j=N}^{\infty} x_j\right) \prod_{j=1}^{N-1} (1 + x_j |z||w|)$$

for all $N \in \mathbb{N}$ and

$$2a|z||w| \leq \frac{1}{s}|w|^2 + a^2s|z|^2$$

for all $s > 0$, we see that $G_T(z, w) \in \mathcal{F}_{(\beta, \gamma)}$ for all $\beta > a^2s$ and $\gamma > 1/s$. Hence $T : \mathcal{F}_c \rightarrow \mathcal{F}_d$ whenever $c > 0$ and $d > a^2c$.

Theorem 3.4. *Let $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[[z_1, \dots, z_m]]$ be a linear operator such that $G_T(z, w) \in \mathcal{F}_{\beta \oplus \gamma}$. Then T defines a bounded operator $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ for all $\alpha \leq \gamma^{-1}$:*

$$\|T(f)\|_\beta \leq \|G_T(z, \alpha w)\|_{\beta \oplus \alpha} \|f\|_\alpha. \quad (3.4)$$

Moreover T has the integral representation

$$T(f)(z) = \int_{\mathbb{C}^n} f(w) G_T(z, \alpha \bar{w}) d\sigma_\alpha(w). \quad (3.5)$$

Conversely if $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ is a bounded operator, then $G_T(z, w) \in \mathcal{F}_{\beta \oplus \gamma}$ for all $\gamma \gg \alpha^{-1}$.

Proof. Suppose that $H(z, w) \in \mathcal{F}_{\beta \oplus \alpha}$, and define a linear operator by

$$T(f)(z) = \int_{\mathbb{C}^n} f(w) H(z, \bar{w}) d\sigma_\alpha(w).$$

This is well defined since $H(z_0, w) \in \mathcal{F}_\alpha$ for each $z_0 \in \mathbb{C}^n$, and then by Cauchy–Schwartz inequality

$$|T(f)(z)| \leq \|f\|_\alpha \|H(z, \cdot)\|_\alpha. \quad (3.6)$$

It follows from (3.2) that

$$H(z, w) = T(e_\alpha(z, w)) = \sum_{\eta \in \mathbb{N}^n} T(z^\eta) \frac{w^\eta \alpha^\eta}{\eta!} = G_T(z, \alpha w).$$

By (3.6)

$$\|T(f)\|_\beta^2 \leq \|f\|_\alpha^2 \int_{\mathbb{C}^m} \|H(z, \cdot)\|_\alpha^2 d\sigma_\beta(z) = \|f\|_\alpha^2 \|H\|_{\beta \oplus \alpha}^2 < \infty.$$

Hence $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ is a bounded operator.

To prove the first part of the theorem we want to determine when $G_T(z, \alpha w) \in \mathcal{F}_{\beta \oplus \alpha}$ given that $G_T(z, w) \in \mathcal{F}_{\beta \oplus \gamma}$. However $G_T(z, \alpha w) \in \mathcal{F}_{\beta \oplus \alpha^2 \gamma}$ which implies $G_T(z, \alpha w) \in \mathcal{F}_{\beta \oplus \alpha}$ whenever $\alpha^2 \gamma \leq \alpha$ from which the first part of the theorem follows.

Conversely suppose that $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ is a bounded operator so that $\|T(f)\|_\beta \leq C\|f\|_\alpha$ for all $f \in \mathcal{F}_\alpha$ and some $C > 0$. For each fixed $w \in \mathbb{C}^n$ define an entire function in z by $H(z, w) = T(e_\alpha(z, w))$. Now $H(z, w)$ defines an entire function on \mathbb{C}^{n+m} as can be seen as follows. Let $E_k(z, w) = \prod_{j=1}^n (1 + \alpha_j z_j w_j / k)^k$, then

$$\begin{aligned} \|T(E_k(z, w))\|_\beta^2 &\leq C^2 \|E_k(z, w)\|_\alpha^2 \leq C^2 \|E_k(|z|, |w|)\|_\alpha^2 \\ &\leq D \int_{\mathbb{C}^n} \exp \left(- \sum_{j=1}^n \alpha_j (|z_j|^2 - 2|z_j||w_j|) \right) dm = K(|w_1|, \dots, |w_n|) < \infty. \end{aligned}$$

By (3.3) $T(E_k(z, w))$ is locally bounded, so by Vitali's theorem $T(E_k(z, w)) \rightarrow H(z, w)$ uniformly on compact subsets of \mathbb{C}^{n+m} . Hence $H(z, w) = T(e_\alpha(z, w)) =$

$G_T(z, \alpha w)$ is an entire function. Now let $\alpha' \gg \alpha$, then

$$\begin{aligned} \|H(z, w)\|_{\beta \oplus \alpha'}^2 &= \int_{\mathbb{C}^n} \|H(\cdot, w)\|_{\beta}^2 d\sigma_{\alpha'}(w) = \int_{\mathbb{C}^n} \|T(e_{\alpha}(z, w))\|_{\beta}^2 d\sigma_{\alpha'}(w) \\ &\leq C^2 \int_{\mathbb{C}^n} \|e_{\alpha}(z, w)\|_{\alpha}^2 d\sigma_{\alpha'}(w) = C^2 \|e_{\alpha}(z, w)\|_{\alpha \oplus \alpha'}^2 \\ &= C^2 \prod_{j=1}^n \frac{\alpha'_j}{\alpha'_j - \alpha_j}. \end{aligned}$$

Hence $H(z, w) = G_T(z, \alpha w) \in \mathcal{F}_{\beta \oplus \alpha'}$ for all $\alpha' \gg \alpha$, from which the theorem follows. \square

Proof of Theorem 3.1. By Theorems 2.1 and 3.4, and (3.1) it remain to prove (2). Let $f = \sum_{\gamma} a_{\gamma} z^{\gamma} \in \mathcal{L}\text{-}\mathcal{P}_{\alpha}(\mathbb{C})$, and let

$$f_k(z) = \sum_{\gamma \leq \mathbf{k}} \frac{(\mathbf{k})_{\gamma}}{\mathbf{k}^{\gamma}} a_{\gamma} z^{\gamma},$$

where $\mathbf{k} = (k, \dots, k) \in \mathbb{N}^n$. By Lemma 2.4 f_k is stable or identically zero for all k . Since $(\mathbf{k})_{\gamma}/\mathbf{k}^{\gamma} \leq 1$ for all k and γ and $(\mathbf{k})_{\gamma}/\mathbf{k}^{\gamma} \rightarrow 1$ as $k \rightarrow \infty$ for all $\gamma \in \mathbb{N}^n$ we have $f_k \rightarrow f$ in \mathcal{F}_{α} . Since T is bounded $T(f_k) \rightarrow T(f)$ in \mathcal{F}_{β} , and hence $T(f_k) \rightarrow T(f)$ uniformly on compact subsets of \mathbb{C}^n . Now $T(f_k) \in \mathcal{L}\text{-}\mathcal{P}_{\beta}(\mathbb{C})$ for all k since T is a Laguerre–Pólya preserver, and the theorem follows. \square

The operators in question are bounded, so the dual operator is well-defined and bounded. Since $T : \mathcal{F}_{\alpha} \rightarrow \mathcal{F}_{\beta}$ and $T^* : \mathcal{F}_{\beta} \rightarrow \mathcal{F}_{\alpha}$ are related by $\langle T(f), g \rangle_{\beta} = \langle f, T^*(g) \rangle_{\alpha}$, we see that

$$G_{T^*}(w, \beta v) = \overline{G_T(\overline{v}, \overline{\alpha w})}, \quad (3.7)$$

by setting $f(z) = e_{\alpha}(z, \overline{w})$ and $g(z) = e_{\beta}(z, \overline{v})$ and using (3.2). As in Remark 2.6 we obtain:

Corollary 3.5. *Let $T : \mathcal{F}_{\alpha} \rightarrow \mathcal{F}_{\beta}$ be a bounded linear operator of rank at least two (or at least three if T is considered as acting on real entire functions). Then T is a Laguerre–Pólya preserver if and only if its dual $T^* : \mathcal{F}_{\beta} \rightarrow \mathcal{F}_{\alpha}$ is a Laguerre–Pólya preserver.*

Example 3.6. Let T be a differential operator with constant coefficient, $T = g(\partial/\partial z)$, where $\partial/\partial z = (\partial/\partial z_1, \dots, \partial/\partial z_n)$ and g is an entire function. For which $\gamma, \beta \in \mathbb{R}_+^n$ is $T : \mathcal{F}_{\alpha} \rightarrow \mathcal{F}_{\gamma}$ a bounded operator? This was answered in [8, Proposition 2.5], and we shall see how it may be derived from Theorem 3.4.

The symbol of T is $g(w)e^{z \cdot w}$, and $\|e^{z \cdot w}\|_{\beta \oplus \eta}^2 = \prod_{j=1}^n (\beta_j \eta_j / (\beta_j \eta_j - 1)) < \infty$ if and only if $\beta \gg 1/\eta$. Hence if $\alpha \in \mathbb{R}_+^n$ and

$$M_{\alpha}(g) := \sup_{z \in \mathbb{C}^n} \left[\exp \left(- \sum_{j=1}^n \alpha_j |z_j|^2 / 2 \right) |g(z)| \right] < \infty,$$

then

$$\begin{aligned} \|g(w)e^{z \cdot w}\|_{\beta \oplus (\eta + \alpha)}^2 &= C^2 \int_{\mathbb{C}^{2n}} |g(w)e_{\alpha/2}(-w, \overline{w})|^2 |e^{z \cdot w}|^2 d\sigma_{\beta \oplus \eta}(z, w) \\ &\leq C^2 M_{\alpha}(g)^2 \|e^{z \cdot w}\|_{\beta \oplus \eta}^2, \end{aligned}$$

where $C > 0$ is a constant. By Theorem 3.4, $T : \mathcal{F}_{1/(\eta+\alpha)} \rightarrow \mathcal{F}_\beta$ whenever $\beta \gg 1/\eta$. Hence $T : \mathcal{F}_\gamma \rightarrow \mathcal{F}_\beta$ whenever $\gamma \ll 1/\alpha$ and $\beta \gg \gamma/(1-\alpha\gamma)$, where 1 is the all ones vector.

This is sharp by Theorem 3.4, which can also be seen from the following example of Pólya: Let $g(z) = \exp(az/2)$ and $f(z) = \exp(bz/2)$ where $a, b > 0$. Then $M_\alpha(g) < \infty$ if and only if $\alpha \geq a$ and $f \in \mathcal{F}_\gamma$ if and only if $\gamma > b$. Hence if $ab < 1$, then by the above, $g(d/dz)f \in \mathcal{F}_\beta$ for any $\beta > b/(1-ab)$. Now

$$g(d/dz)f = \frac{1}{\sqrt{1-ab}} \exp\left(\frac{1}{2} \frac{b}{1-ab} z^2\right),$$

so that $g(d/dz)f \in \mathcal{F}_\beta$ if and only if $\beta > b/(1-ab)$.

4. LEE-YANG THEOREMS

The Lee–Yang theorem and its extensions assert that the Fourier–Laplace transform of Gibbs measures of various spin models are nonzero whenever all variables are in the open right half-plane, and they serve as important tools in the rigorous study of phase transitions in lattice spin systems [12]. We follow the approach to the Lee–Yang theorem developed by Lieb and Sokal [8] that uses linear operators preserving non-vanishing properties. For another successful method which uses Asano contractions we refer to [12, 13] and the references therein.

Denote by $\mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$ the space of all entire functions in n variables that are uniform limits on compact subsets of \mathbb{C}^n of polynomials that are nonvanishing whenever all variables are in the open right half-plane of the complex plane. Thus $f(z_1, \dots, z_n) \in \mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$ if and only if $f(-iz_1, \dots, -iz_n) \in \mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$. A measure μ on \mathbb{R}^n is said to have the *Lee–Yang property* if its Fourier–Laplace transform

$$\hat{\mu}(w) := \int_{\mathbb{R}^n} e^{z \cdot w} d\mu(z)$$

is an entire function in $\mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$. More generally a continuous linear functional $\phi : \mathcal{F}_\beta \rightarrow \mathbb{C}$ has the Lee–Yang property if the map $\hat{\phi}$ defined by $w \mapsto \phi(e^{z \cdot w})$ defines an entire function in $\mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$. It is natural to extend the definition to linear operators: A bounded linear operator $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ has the Lee–Yang property if $T(e^{z \cdot w}) = G_T(z, w) \in \mathcal{L}\text{-}\mathcal{Y}_{n+m}(\mathbb{C})$.

Example 4.1. Here are a few basic examples of measures on \mathbb{R} with the Lee–Yang property:

- (1) If $\mu = (\delta_a + \delta_b)/2$, where δ_a and δ_b are the Dirac measures centered at $a, b \in \mathbb{R}$, then

$$\hat{\mu}(z) = \exp\left(\frac{a+b}{2}z\right) \cosh\left(\frac{a-b}{2}z\right).$$

Hence $\mu : \mathcal{F}_c \rightarrow \mathbb{C}$ has the Lee–Yang property for all $c > 0$ and $a + b \geq 0$. Moreover if a and b are allowed to be non-real, then μ has the Lee–Yang property if and only if $a - b \in \mathbb{R}$ and $\operatorname{Re}(a + b) \geq 0$.

- (2) If μ is Lebesgue measure on the interval $[a, b]$, then

$$\hat{\mu}(z) = \int_a^b e^{zx} dx = \frac{2}{z} \exp\left(\frac{b-a}{2}z\right) \sinh\left(\frac{a+b}{2}z\right).$$

Hence $\mu : \mathcal{F}_c \rightarrow \mathbb{C}$ has the Lee–Yang property for all $c > 0$.

(3) If $d\mu(x) = e^{-bx^2/2}dx$ on \mathbb{R} with $b > 0$, then

$$\hat{\mu}(z) = \int_{\mathbb{R}} e^{zx} e^{-bx^2/2} dx = \sqrt{2\pi/b} \exp\left(\frac{z^2}{2b}\right).$$

Hence $\mu : \mathcal{F}_a \rightarrow \mathbb{C}$ has the Lee–Yang property for all $a < b$.

The results developed in the previous sections apply (by a change of variables) to *Lee–Yang preservers* which we define to be linear operators $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[[z_1, \dots, z_m]]$ that map $\mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C}) \cap \mathbb{C}[z_1, \dots, z_n]$ into $\mathcal{L}\text{-}\mathcal{Y}_m(\mathbb{C})$. For $\beta \in \mathbb{R}_+^n$, let $\mathcal{L}\text{-}\mathcal{Y}_\beta(\mathbb{C}) = \mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C}) \cap \mathcal{F}_\beta$.

Theorem 4.2. *Let $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[[z_1, \dots, z_m]]$ be a linear operator of rank at least two. Then T is a Lee–Yang preserver if and only if T has the Lee–Yang property, that is, $G_T(z, w) \in \mathcal{L}\text{-}\mathcal{Y}_{\beta \oplus \gamma}(\mathbb{C})$ for some $\beta \in \mathbb{R}_+^n$ and $\gamma \in \mathbb{R}_+^m$.*

Moreover if $G_T(z, w) \in \mathcal{L}\text{-}\mathcal{Y}_{\beta \oplus \gamma}(\mathbb{C})$, then

- (1) *T extends to a bounded linear operator $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ of the form (3.4) and (3.5) for all $\alpha \leq \gamma^{-1}$, and*
- (2) *$T : \mathcal{L}\text{-}\mathcal{Y}_\alpha(\mathbb{C}) \rightarrow \mathcal{L}\text{-}\mathcal{Y}_\beta(\mathbb{C})$, for all $\alpha \leq \gamma^{-1}$.*

Corollary 4.3. *Let $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ be a bounded linear operator of rank at least two. Then T is a Lee–Yang preserver if and only if its dual $T^* : \mathcal{F}_\beta \rightarrow \mathcal{F}_\alpha$ is a Lee–Yang preserver.*

Remark 4.4. If $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ is a bounded linear operator, z'_1, \dots, z'_k are new variables and $\gamma \in \mathbb{R}_+^k$, then T extends to a bounded linear operator $\tilde{T} : \mathcal{F}_{\alpha \oplus \gamma} \rightarrow \mathcal{F}_{\beta \oplus \gamma}$ by setting $\tilde{T}(f(z, z')) = T(f(z, z'))$ and where T only acts on the z -variables. Note also that the symbol of \tilde{T} is $e^{z' \cdot w'} T(e^{z \cdot w})$ so that T has the Lee–Yang property if and only if \tilde{T} has the Lee–Yang property.

The next theorem shows that bounded linear operators with the Lee–Yang property are closed under composition. This can be seen as an ultimate generalization of [8, Proposition 2.9] and serves as a fundamental tool to prove Lee–Yang theorems for one-component models.

Theorem 4.5. *Suppose that $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ and $S : \mathcal{F}_\beta \rightarrow \mathcal{F}_\gamma$ have the Lee–Yang property. Then so does $S \circ T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\gamma$.*

In particular, if $\phi : \mathcal{F}_\beta \rightarrow \mathbb{C}$ and $T : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\beta$ have the Lee–Yang property, then $\phi \circ T : \mathcal{F}_\alpha \rightarrow \mathbb{C}$ has the Lee–Yang property.

Proof. The symbol of $S \circ T$ is $\tilde{S}(G_T(z, w))$, where $\tilde{S} : \mathcal{F}_{\beta \oplus \kappa} \rightarrow \mathcal{F}_{\gamma \oplus \kappa}$ is as in Remark 4.4. Now $G_T(z, w) \in \mathcal{L}\text{-}\mathcal{Y}_{\beta \oplus \kappa}$ for all $\kappa \gg \alpha^{-1}$. By Theorem 4.2 and Remark 4.4 \tilde{S} is a Lee–Yang preserver and thus $\widehat{S \circ T} = \tilde{S}(G_T(z, w)) \in \mathcal{L}\text{-}\mathcal{Y}_{\gamma \oplus \kappa}$. \square

The following corollary is an equivalent formulation of the most general (formal) one component Lee–Yang theorem in [8] from which many others follow.

Corollary 4.6. *Suppose that $\phi : \mathcal{F}_\beta \rightarrow \mathbb{C}$ has the Lee–Yang property. If $\alpha, \beta, \gamma \in \mathbb{R}_+^n$ satisfy $\alpha + \gamma \leq \beta$, and $g \in \mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$ satisfies $M_\alpha(g) < \infty$, then $\psi : \mathcal{F}_\gamma \rightarrow \mathbb{C}$ defined by $\psi(f) = \phi(fg)$ has the Lee–Yang property.*

Proof. Clearly the operator $T(f) = fg$ is a Lee–Yang preserver. By Theorem 4.5 it remains to prove that $T : \mathcal{F}_\gamma \rightarrow \mathcal{F}_{\alpha+\gamma}$ is a bounded operator. If $f \in \mathcal{F}_\gamma$ and $M_\alpha(g) < \infty$, then

$$\begin{aligned} \|gf\|_{\alpha+\gamma}^2 &= \prod_{j=1}^n (1 + \alpha_j/\gamma_j) \int_{\mathbb{C}^n} |g(z)e_{\alpha/2}(-z, \bar{z})|^2 |f(z)|^2 d\sigma_\gamma(z) \\ &\leq \prod_{j=1}^n (1 + \alpha_j/\gamma_j) M_\alpha(g)^2 \|f\|_\gamma^2. \end{aligned}$$

□

Since $e^{z_1 z_2}, e^{z_1^2} \in \mathcal{L}\text{-}\mathcal{Y}_2(\mathbb{C})$, we see that $e_J(z) = \exp(\sum_{i,j=1}^n J_{ij} z_i z_j) \in \mathcal{L}\text{-}\mathcal{Y}_n(\mathbb{C})$ for all matrices J such that $J_{ij} \geq 0$ for all j . The original Lee–Yang theorem [6] states that measures of the type

$$d\mu = e_J(z) d\mu_1(z_1) \cdots d\mu_n(z_n)$$

where μ_1, \dots, μ_n are measures on \mathbb{R} as in Example 4.1 (1) with $a = -b = 1$ and J is a (entry-wise) nonnegative symmetric matrix. Clearly the direct product of two measures with the Lee–Yang property has the Lee–Yang property, so the original Lee–Yang theorem follows from Corollary 4.6.

Newman’s Lee–Yang theorem [9] asserts (in our language) that

$$d\mu = e_J(z) d\mu_1(z_1) \cdots d\mu_n(z_n)$$

has the Lee–Yang property whenever μ_1, \dots, μ_n are even measures on \mathbb{R} with the Lee–Yang property and J is a nonnegative symmetric matrix such that $\hat{\mu} \in \mathcal{F}_\beta$ for some $\beta \in \mathbb{R}_+^n$. Hence Newman’s theorem also follows from Corollary 4.6.

The Lee–Yang theorem of Lieb and Sokal [8, Theorem 3.2] asserts that

$$d\mu = e_J(z) d\mu_0$$

has the Lee–Yang property whenever μ_0 has the Lee–Yang property, J is a nonnegative symmetric matrix, and $\hat{\mu} \in \mathcal{F}_\beta$ for some $\beta \in \mathbb{R}_+^n$. Hence this theorem also follows from Corollary 4.6.

Remark 4.7. To apply Newman’s or Lieb and Sokal’s theorem one needs to know for which symmetric matrices A with nonnegative entries $\exp\left(\sum_{i,j} A_{ij} z_i z_j\right) \in \mathcal{F}_\beta$, so that one can use Corollary 4.6. This happens if and only if

$$\sum_{i,j} A_{ij} |z_i| |z_j| \leq \sum_i \alpha_i |z_i|^2$$

for some $\alpha \ll \beta/2$, that is, if and only if $\|D_\alpha^{-1/2} A D_\alpha^{-1/2}\| \leq 1$ for some $\alpha \ll \beta/2$, where $\|\cdot\|$ denotes the operator norm and $D_\alpha = \text{diag}(\alpha_1, \dots, \alpha_n)$ is a diagonal matrix.

5. AN OPEN PROBLEM

We end by generalizing an important open problem from [8]. Let $\Gamma \subset \mathbb{R}^n$ be an open convex cone, and let $\mathcal{P}_n(\Gamma)$ be the set of polynomials in n variables that are non-vanishing whenever the real parts of the variables are in Γ .

Problem 5.1. *Let $\Gamma \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^m$ be two open convex cones. Characterize all linear operators $T : \mathbb{C}[z_1, \dots, z_n] \rightarrow \mathbb{C}[z_1, \dots, z_m]$ such that $T(\mathcal{P}_n(\Gamma)) \subseteq \mathcal{P}_m(\Lambda) \cup \{0\}$.*

When $\Gamma = \mathbb{R}_+^n$ and $\Lambda = \mathbb{R}_+^m$, Problem 5.1 is just Theorem 2.1 (by a rotation of the variables). A solution to Problem 5.1 would entail optimal Lee–Yang theorems for N -component models when $N \geq 3$, see [8, Section 5] where partial results on Problem 5.1 for differential operators in the Weyl-algebra were obtained. Progress on Problem 5.1 would also be interesting for the convex optimization community, see e.g. [11], since a homogeneous polynomial P is in $\mathcal{P}_n(\Gamma)$ if and only if P is hyperbolic with hyperbolicity cone containing Γ . Thus Problem 5.1 asks how one may deform a hyperbolic polynomial and retain hyperbolicity.

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